INVENTIONAL GEOMETRY

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CONTENTS

PREFACE

This little book is written for the grammar school. The entire work may be completed in the last two years of the course, with two or three lessons each week of forty minutes each. The exercises are carefully graded as to difficulty and arranged in the order of logical sequence.

For a *one-year course* of two or three lessons each week, omit the following exercises: 4, 8–16, 34–42, 45–47, 52, 53, supplementary and complementary angles, 57–66, 73–83, 87–90, 100–103, 107, 108, 130, 131, 136–139, 145–156, 171–184, 186–193, 196–198, 211–220, 228–233, 236–239, 241–252, 264, 266, 291–299, 307–309, 311, 330, 331, 337, 343, 349, 350, 353, 357–363, 369–372, 376–386. The remaining part of the work makes a one-year course of study strictly logical and complete in itself.

The amount of work to be accomplished by any class must depend upon the time assigned, the age and ability of the pupils, and other modifying circumstances. The teacher will readily adjust the work to meet local requirements.

For reasons given in the "Introduction," the amount of Euclidian geometry given is a minimum. Solid geometry is omitted because it is comparatively a poor field for inventional work. Similar polygons are omitted because the subject involves many principles of geometry beyond the ability of grammar-school pupils to comprehend. The simple problem of finding the height of a tall object by comparing the length of its shadow with the length of the shadow of a vertical staff whose height is known may be given; but beyond this in mensuration these young people cannot well go.

In the "Introduction" the author gives a full discussion of the principles upon which this book is written, together with some suggestions on the method of teaching "Inventional Geometry."

Suggestions or criticisms from all members of the teaching profession will be gratefully acknowledged.

RICHMOND HILL,
NEW YORK CITY.

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INTRODUCTION

GEOMETRY is introduced in the grammar school for three reasons: First, it gives much useful and practical knowledge; second, it prepares pupils for the formal study of demonstrative geometry in the high school; third, it has a most desirable disciplinary value.

The practical value of this science is familiar to all. Says William George Spencer, the father of Herbert Spencer: "When it is considered that by geometry the architect constructs our buildings, the civil engineer our railways; that by a higher kind of geometry, the surveyor makes a map of a country or of a kingdom; that a geometry still higher is the foundation of the noble science of the astronomer, who by it not only determines the diameter of the globe he lives upon, but as well the sizes of the sun, moon, and planets, and their distances from us and from each other; when it is considered, also, that by this higher kind of geometry, with the assistance of a chart and a mariner's compass, the sailor navigates the ocean with success, and thus brings all nations into amicable intercourse—it will surely be allowed that its elements should be as accessible as possible."

A PREPARATION FOR HIGH-SCHOOL GEOMETRY

Says Professor George A. Hill: "The formal method of teaching Geometry, as we find it in Euclid and in the common text-books, whatever may be its merits, is a very bad method for beginners. It makes the study of the subject unnecessarily dry, tedious, and difficult; and it ignores, in most cases, the great law of mental development, that clear perceptions and intuitions must precede the intelligent use of the faculties of comparison and reasoning."

All who have taught geometry in the high school know how slowly and with what difficulty beginners take up the subject. The chief reason for this is that the beginner has great difficulty in fully grasping the geometric concepts. A preliminary training in concrete geometry removes this difficulty.

Inventional geometry bears the same relation to Euclid that nature studies in the grades bear to science. It is a preliminary preparation in which pupils are required to make accurate use of technical terms, and in which they are to learn nothing which must be unlearned in the high school.

This second reason for the study of geometry in the elementary school is incidental only. Alone, it is not a sufficient warrant for such introduction; because a course of study for the elementary school, and for the high school as well, should be so formed that pupils, whenever they may leave these schools, shall have received the best, the very best, that we can give them. It is our solemn duty to care for the masses. Let the high school and the university care for themselves.

A DISCIPLINARY STUDY

The third and fundamental reason for introducing geometry in the elementary school is the mental discipline which it yields. And is not this the overshadowing reason for which we teach geometry in the high school as well? The study of Euclid is a preparation for the pursuit of other and higher branches of mathematics, such as trigonometry, surveying and navigation, astronomy, etc. But this preparation is only incidental to the study of Euclid. And so we teach inventional geometry to grammar-school pupils for the mental training which they derive from it. Estimated from this standpoint alone, it is superior to all other subjects taught in the elementary school.

In the years which are past, we relied upon arithmetic for the training of the child's analytic powers. Especially in the olden time, when the now graybeards were school-boys, the so-called mental arithmetic, with its formidable array of miscellaneous and intricate problems, served this purpose well. This value of the study of arithmetic, especially as taught to-day, is vastly overestimated, and for this reason: the work in arithmetic, and in algebra as well, is largely imitative. It is work by rule. Thus, "To divide one fraction by another, invert the divisor and proceed as in multiplication." Now the analysis which brings out the reasons for the process is a valuable exercise; but the formal working of a long list of examples by an established rule is a mechanical imitation and gives no analytic power whatever. Not so in the geometry; for in geometry no two problems are solved by the same

rule; in geometry every step is a challenge to the pupil's power of analysis.

Neither arithmetic nor algebra gives that rigorous training in sharp thinking so desirable. The inventional geometry alone remedies this defect in our elementary education by its severe training in clear, consecutive, logical thought. It likewise trains the eye to quick and accurate perception, and the hand to great precision in drawings and measurements. It forms the habit of concentrated thought, cultivates a love for the beautiful in construction and design, and creates a keen observation through the consideration of the manifold geometric forms everywhere abounding in nature and in the mechanical arts. It gives correct ideas of magnitude and form; and because of the absolute precision of its data, it trains the mind to think clearly, to reason accurately, and to judge correctly. But more than all these, the grammar-school geometry develops the power of invention, a faculty always dormant in youth and heretofore almost totally ignored. In this the inventional geometry is unique. The fundamental object of this science is to cultivate the power of invention, to stimulate the mathematical imagination, and to develop the mathematical perceptions of the child through his own observation and discovery.

The treatment of the subject as set forth in the following pages is based upon five principles, as follows:

1. *Grammar-school geometry is inventional, not demonstrative; it deals with the concrete, not with the abstract.*

Inventional geometry deals with the concrete always. Frequently pupils are required to generalize, but not

until they have been led up to the generalization through the concrete. In the kindergarten young children rapidly acquire correct ideas of number and form from objects. The student learns more chemistry from six weeks' work in the laboratory than by memorizing the whole of his text-book. Likewise pupils rapidly acquire the properties of magnitude by observation and experiment, by direct measurement, by cutting and folding, by superposition. It is not sufficient that the student be told that two parts of hydrogen and one of oxygen unite to form water. He has no real knowledge of that fact until he has verified it by experiment in the laboratory. Likewise pupils must with their own hands superpose one magnitude upon the other, see with their own eyes that they coincide at every point, and conclude with their own judgment that magnitudes under given conditions are equal.

Consider the following exercise: "Draw an isosceles triangle and measure its base angles. Repeat the experiment in different isosceles triangles. Generalize." The pupil draws an isosceles triangle, measures the base angles with the protractor, and finds them equal. He sees that the base angles in that particular triangle are equal. He repeats the measurement in different isosceles triangles, and in each case finds the same relation, when the general truth is forced upon his mind that "In any isosceles triangle the base angles are equal." The general conclusion is drawn from particular cases. In other words, the pupil first acquires individual notions from which his mind gradually passes to general notions. Now observe the difference in method. In the high-school geometry the pupil is told

the general truth which he is required to prove by a process of pure reasoning. In the inventional geometry the pupil by concrete exercises is led to find out the truth himself. He thus makes his own discovery. He thus acquires real knowledge. In this way pupils are to learn all the facts of geometry which this work makes it necessary for them to know. At every step of the way there must be concrete illustration. This is the secret of success in teaching geometry in the grammar school.

2. *A knowledge of geometry is subordinate to mental discipline.*

Thus far we have spoken of the method of developing certain truths found in Euclid. But this is only preliminary to the real work of our science. These truths of geometry are only the tools with which pupils work to aid them in their invention. We teach only a very small part of Euclid. We select only those theorems and problems which can best be established by observation and experiment, and upon which we can build the most desirable inventional exercises. Let it be distinctly understood that by this study we aim to get a certain kind of mental discipline which no other study gives; that to accomplish this end we make use of some elementary truths found in Euclid, selected to suit our purpose; and that we make these truths the basis of our science for the reason that they suit our purpose better than any other facts available. Look well to the intellectual culture of the pupils; their knowledge of geometry will take care of itself.

But some will say: "Why take time in developing our data? Why not give the pupils these basic truths

of Euclid outright and hasten on?" I will tell you why. These truths of Euclid constitute the foundation upon which we build our science, and pupils must grasp them firmly. Now it is a well-established law of mental growth that facts are best learned by experience. To-day you tell your pupils that the base angles of an isosceles triangle are equal, and to-morrow they have forgotten it. Lead them into the discovery of the fact in the manner that I have described, and they will remember it forever. All real knowledge, all vivid images, all clear concepts, are acquired through experience. For the masses clear thinking is impossible without concrete illustration. Moreover, we are thus consistent in our philosophy, for the chosen facts of geometry which we make the basis of our science have been themselves established by the inventional method.

Briefly this second principle may be stated as follows: Clear geometric concepts are more desirable than many obscure ideas. Instruction which develops strength without knowledge is better than instruction which develops knowledge without strength. Instruction which develops both knowledge and strength is ideal instruction.

3. *Formal definitions should not be memorized.*

Concise mathematical definitions are abstract things, and hence have no place in this concrete science; they are of great importance in demonstrative geometry, because they constitute a part of the data upon which proofs are built up. Tax pupils with the task of memorizing formal definitions set in technical terms which have no meaning to them, and you make the subject irksome to them. The teacher should explain the

meaning of technical terms and the import of definitions in a concrete way. Afterward pupils should be required to illustrate these definitions by means of accurately constructed figures as an evidence that they understand them. When called upon, they should be able instantly to draw a circle, a chord, a tangent, a parallelogram, a trapezoid, etc. This is evidence that they understand these terms and that they have vivid mental images of these concepts.

4. *The keenest incentive to intellectual exertion is the consciousness of personal achievement.*

The pupils should do the work, not the teacher. They should have but little assistance. They do not need it. All they require is proper guidance. In this study we are setting out upon a voyage of discovery. We are to acquire knowledge by observation, by construction, and always in the concrete. Let the pupils seek the truth. The teacher by skilful encouragement may lead them into the discovery, but she must not make the discovery. Let the pupils make their own observations, invent their own constructions, and draw their own conclusions. The teacher who begins by solving the problems for the pupils must continue to solve them to the end. Such teaching is worse than useless. It begets weakness, inattention, loose habits of thought, indistinct ideas, and finally a hatred for the subject.

Pupils should be required to solve no problem before they have had all the data necessary to its solution. For example, there are three fundamental constructions in Euclidian geometry of great importance; viz., To draw a perpendicular to a given line: first, at a given

point in the line; second, from a given point without the line; third, at the end of the line. Now it is necessary that pupils draw these perpendiculars early in the course, before it is possible to give them the necessary data for their construction with ruler and compasses. Accordingly they draw them with ruler and right triangle; but later, just as soon as they have had the necessary data, they are required to draw them in the classic way.

In order that the pupils be not intimidated and may become self-reliant, the work should begin with the simplest kind of problems. Exercises should be so graded in difficulty, so arranged in the order of logical sequence, that the pupils by them and by skilful encouragement are led on and on to observe, to think, to discover, to invent, all for themselves. This the following work aims to do. It lays great stress upon the order of logical sequence. Pupils gradually accumulate many facts. But this is not sufficient. These facts must be correlated. Unrelated facts produce confusion and chaos. Pupils must know these facts in their logical relations to each other in order that their reasoning powers may be called into action; in order that they may draw general conclusions; and in order that they may be able to make an intelligent application of these facts in the solution of problems. Their ability to do this is the true measure of their acquired intellectual power.

5. *Enthusiasm is the key-note to success.*

Remember that the primary object of this science is to quicken the mathematical imagination and to develop the power of invention. To do this successfully, the

subject must be made so intensely interesting to the pupils that it becomes a fascination to them, and all unnecessary requirements which make the subject irksome should be studiously avoided. Make this a kindergarten in geometry. When the subject is properly presented through a judicious selection of problems, pupils run wild with enthusiasm. Remember that great inventions are made by those who are intensely enthusiastic over the subject in hand. "The inventive power grows best in the sunshine of encouragement," says Spencer; and he might well have added, it withers and dies in cold indifference.

GENERAL SUGGESTIONS

Apparatus. Each pupil should have a graduated ruler with straight edge, a protractor, a right triangle, and compasses. For blackboard use the class should be provided with the same instruments of suitable size.

All drawings should be made neatly and accurately. This direction is of vital importance and should be insisted upon from the first lesson to the last. Since pupils are to discover geometric truths by direct measurement, it is self-evident that all figures should be constructed with the greatest possible precision. That the base angles of an isosceles triangle should give the same reading on the protractor, the legs of the triangle must be exactly equal. Moreover, an accurately constructed figure frequently suggests the solution of a problem, while a poorly constructed figure generally leads into error.

Pupils should be accurate in the use of language. Careless habits of expression mean obscure thoughts. When

a pupil says, "The angles of an isosceles triangle are equal," something is wrong. Either he has an incorrect idea or he does not correctly express the idea which he has. In either case the teacher should set the pupil right by insisting upon a correct statement of the full idea.

In our courts of justice we frequently hear this forcible injunction: "You do solemnly swear that you will tell the truth, the whole truth, and nothing but the truth." So the pupil should express the idea, the whole idea, and nothing but the idea.

Skilful questioning. The general method by which the teacher leads the pupil successfully through this study consists in skilful questioning. This is exceedingly important. To question skilfully the teacher must have a thorough knowledge of Euclidian geometry. Questions, to be effective, must be clear, pointed, terse, and, above all, they must be put in the order of logical sequence. To do this successfully will tax the teacher's greatest skill. To the art of skilful questioning the teacher should devote her most thoughtful study.

CHAPTER I

GEOMETRIC CONCEPTS

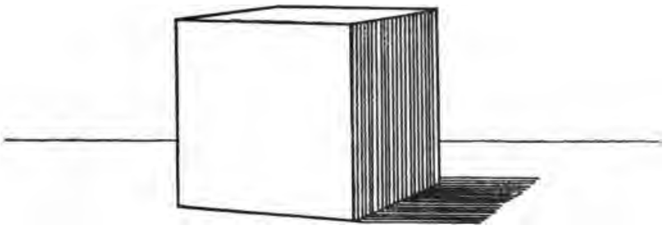

Fig. 1.—A Cube.

A block of wood, as represented in Fig. 1, is called a **body**. This particular body, because of its shape, is called a **cube.**

A cube is bounded by flat faces which we call **surfaces.** If the straight edge of a ruler be placed upon a face of a perfect cube, every point in the edge of the ruler will touch the face of the cube. Such a surface we call a **plane surface,** or simply a **plane.**

The intersection of two faces of a cube forms a **line** (edge).

The intersection of two edges of a cube forms a **point** (corner).

A cube has three dimensions—**length, breadth,** and **thickness.**

A surface has two dimensions only, **length** and **breadth.**

A line has one dimension only, **length.**

A point has no dimension. It has **position** alone.

The ends of a line are *points,* called its *end-points,* or *extremities.*

A point is represented by a dot, designated by a letter, and read by reading the letter. Thus, the point *A* (Fig. 2).

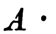

FIG. 2.

FIG. 3.

A line is represented by a mark, designated by letters placed at different points in it, usually its end-points, and read by reading the letters. Thus, the line *AB* (Fig. 3).

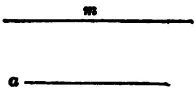

FIG. 4.

We sometimes designate a line by a single letter. Thus, the line *m*, the line *a* (Fig. 4).

1. How many faces has a cube? how many edges? how many corners?

CHAPTER II

STRAIGHT LINES

2. Mark a point on a piece of paper and draw several lines through it. How many lines can be drawn through a point?

Note. The word *line* means a *straight line.*

3. Mark two points on a piece of paper and draw a line through them. Draw a second line through the same two points. How many lines can be drawn through two points? How many points determine (fix in position) a straight line?

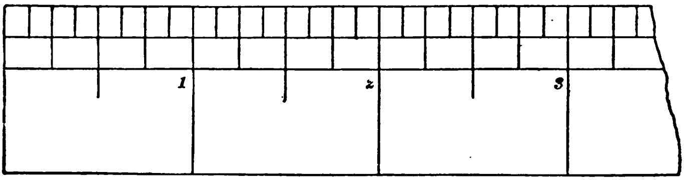

FIG. 5.—Ruler graduated to eighths of an inch.

A **ruler,** or **straight-edge,** is an instrument used for drawing straight lines.

A **graduated ruler** is used for measuring short lines (Fig. 5).

4. Devise an experiment by which you can test your ruler to determine whether or not it has a straight edge.

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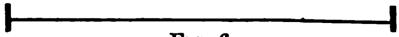


Fig. 6.

5. Draw a line two inches long and cross the end-points as in Fig. 6.

6. Draw a line $2\frac{1}{2}$ inches long; $2\frac{5}{8}$ inches; $3\frac{1}{8}$ inches.

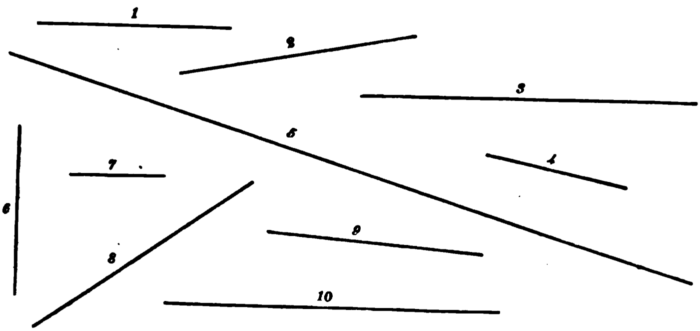


Fig. 7.

7. Measure the lines in Fig. 7, and write on a piece of paper the length of each opposite its number.

8. Measure the length of this page.

9. Measure the length and width of a sheet of paper.

10. Measure the length and width of your desk.

11. Measure your height, the span of your outstretched arms, and the length of your step in walking.

12. Draw a line $\frac{5}{8}$ of an inch long. What distance does it represent if $\frac{1}{8}$ of an inch represents one mile?

13. A map of New York State measures 10 inches long. If the map is drawn to a scale of 30 miles to the inch, what is the length of the State in miles?

14. Draw a line representing 500 miles if $\frac{1}{8}$ of an inch represents 25 miles.

15. Draw a line 800 miles long on a scale of 50 miles to $\frac{1}{4}$ of an inch.

16. Turn to a map of the United States and notice the scale. Estimate the distance from New York City to Buffalo; to Boston; to Chicago; to San Francisco; to New Orleans.

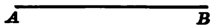


FIG. 8.

17. Draw a line equal to AB (Fig. 8).

Geometric Method. The proper way to take the length of a line is by an instrument called a *pair of dividers*, or simply *dividers* (Fig. 9). This instrument

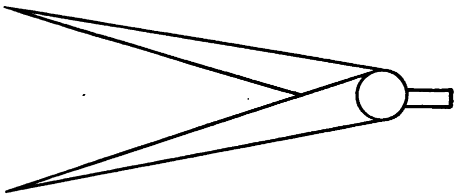


FIG. 9.—Dividers.

has two arms opening on a hinge, each arm ending in a sharp metal point.

To take the length of the line AB, open the dividers until the metal points rest on the points A and B, when the distance between the points of the dividers is the exact length of the line AB. This distance, or length, may now be marked off in any place we choose.

This is the *geometric method* of constructing a line equal to a given line, and should always be used.

Compasses. When one of the metal points of the dividers is replaced by a pencil or pen, the instrument is called a *pair of compasses*, or simply *compasses*.

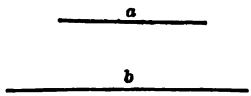


Fig. 10.

18. Construct a line equal to the sum of the lines *a* and *b* (Fig. 10).

19. Construct a line equal to the difference of the lines *a* and *b* (Fig. 10).

20. Draw a line *AB*. Draw two lines *CD* and *EF* each equal to *AB*. Is *CD* equal to *EF*?

21. Two lines can intersect in how many points?

22. In how many points can three lines intersect?

23. In how many points can four lines intersect?

24. Draw three lines so that they can intersect in one point only.

25. How many lines can be drawn joining three points, two by two?

26. How many lines can be drawn joining four points, two by two?

27. How many lines can be drawn joining five points, two by two?

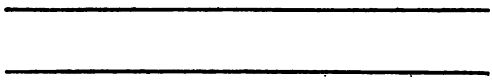


Fig. 11.—Parallel Lines.

Parallel Lines. Lines in the same plane which cannot meet however far produced are called **parallel lines** (Fig. 11).

The lines on ruled paper are parallel.

28. Give illustrations of parallel lines.

Problem. *To draw parallel lines.*

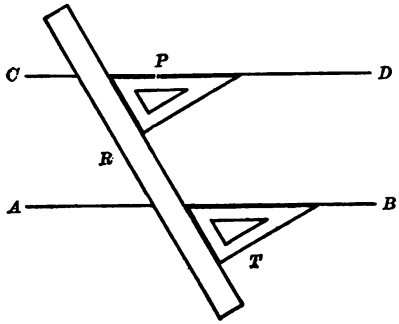


FIG. 12.—Triangle and Ruler.

First Method. Hold the ruler *R* on a piece of paper (Fig. 12), bring one edge of a wooden triangle *T* against it, and draw a line *AB* along a second edge of the triangle. Holding the ruler firmly so that it does not move, slide the triangle along the ruler a short distance and draw a second line *CD* along the same edge of the triangle. *AB* and *CD* are parallel.

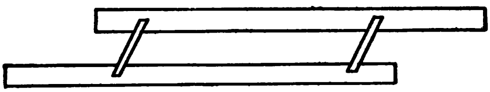


FIG. 13.—Parallel Ruler.

Second Method. The construction of parallel lines is quickly made by a parallel ruler (Fig. 13). This instrument consists of two rulers fastened together by two equal arms moving on pivots at each end. It is so adjusted that the two rulers always remain parallel as they swing on the pivots.

Note. The construction of parallel rulers is an excellent exercise for pupils.

29. Draw two parallel lines one inch apart.

30. Draw three parallel lines one-half inch apart.

31. Draw three lines so that they can intersect in two points only.

32. Draw a line through a given point parallel to a given line.

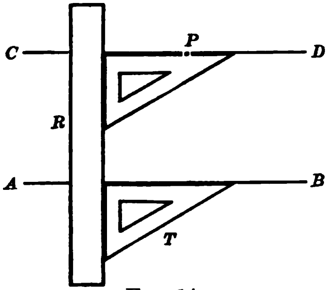


Fig. 14.

(In Fig. 14, *AB* is a given line and *P* a given point. We are required to draw a line through the point *P* parallel to the line *AB*. To do this, bring one edge of the triangle *T* along the line *AB*, and bring the ruler *R* against a second edge of the triangle. Then, holding the ruler firmly, slide the triangle along the ruler until the edge of the triangle falls on *P*. *CD* drawn along the edge of the triangle through *P* is parallel to *AB*.)

33. How many lines can be drawn through a given point parallel to a given line?

CHAPTER III

ANGLES

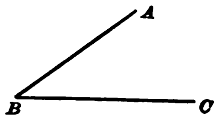


FIG. 15.—An Angle.

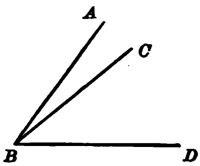


FIG. 16.—Adjacent Angles.

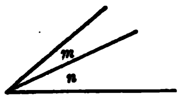


FIG. 17.

When two straight lines are drawn from the same point they form an **angle.** Thus, BA and BC (Fig. 15) drawn from the point B form the angle ABC, or the angle B. BA and BC are the **sides** of the angle, and the point B is the **vertex** of the angle.

Adjacent Angles. When two angles have the same vertex and a common side between them, they are called **adjacent angles.** Thus, CBA and CBD are adjacent angles (Fig. 16). Observe that we read an angle by reading the three letters designating it, and that we read the letter at the vertex between the other two.

For simplicity an angle may be designated by a small italic letter placed between its sides and near its vertex; as, angle m, angle n (Fig. 17).

34. Do you increase the magnitude (size) of an angle by producing its sides?

35. Draw two lines forming one angle.
36. Draw two lines forming two angles.
37. Draw two lines forming four angles.
38. Draw three lines forming two angles.
39. Draw three lines forming three angles.
40. Draw three lines forming five angles.
41. How many angles can you make with three lines?
42. How many angles can you make with four lines?

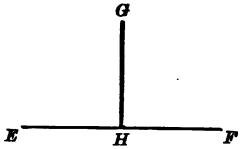


FIG. 18.—Right Angles.

Right Angle. When one straight line meets another making the adjacent angles equal, each is called a **right angle** (Fig. 18). Thus, if the angles *GHE* and *GHF* are equal, each is a right angle.

In this case *GH* and *EF* are called **perpendicular lines.** *GH* is perpendicular to *EF*, and *EF* is perpendicular to *GH.*

The point *H* is called the **foot** of the perpendicular *GH.*

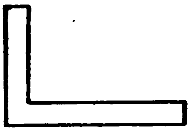


FIG. 19.—The Carpenter's Square.

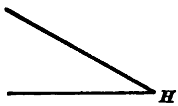


FIG. 20.—Acute Angle.

When the carpenter wishes to form right angles he uses an instrument called the "carpenter's square" (Fig. 19).

Acute Angle. An angle less than a right angle is called an **acute angle**; as, angle *H* (Fig. 20).

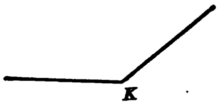


FIG. 21.— Obtuse Angle.

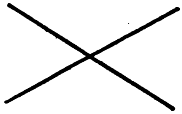


FIG. 22.—Oblique Angles and Oblique Lines.

Obtuse Angle. An angle greater than a right angle (and less than two right angles) is called an **obtuse angle,** as, angle *K* (Fig. 21).

Acute and obtuse angles are called **oblique angles,** and lines forming oblique angles are called **oblique lines** (Fig. 22).

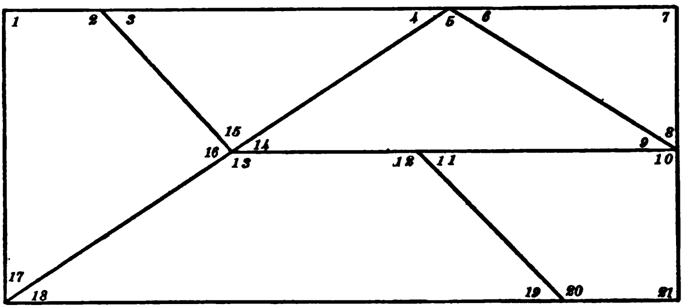


FIG. 23.

43. Describe the different angles in Fig. 23.

44. Point out some right angles in the construction of your school-room.

Can you find some acute and some obtuse angles in the construction of your desk?

45. How long does it take the minute-hand of a watch to describe one right angle?

46. How long does it take the hour-hand of a watch to describe one right angle?

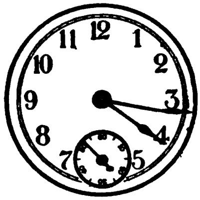


FIG. 24.—A Watch.

47. How many right angles does the minute-hand of a watch describe in one hour?

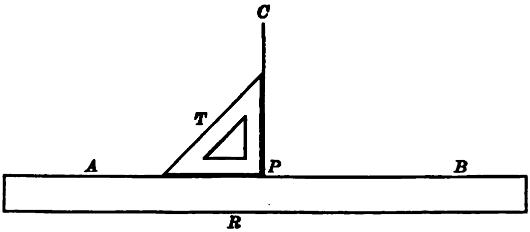


FIG. 25.—Right Triangle and Ruler.

The **right triangle** is a three-cornered piece of wood having one square corner or right angle (Fig. 25). The right triangle T has three straight edges and is used to draw *perpendiculars*, and hence *right angles*. Thus, to draw a perpendicular to the line AB at P, we place one edge of the ruler R along the line AB; we next bring one edge of the triangle T against the ruler, making the vertex of the right angle coincide with the point P; one edge of the triangle is now perpendicular to AB at P; hence a line CP drawn along this edge of the triangle is a perpendicular to AB at P, and the angles CPA and CPB are right angles.

48. With the ruler and triangle draw a perpendicular to a given line at its end-point.

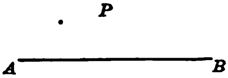


Fig. 26.

Fig. 27.

49. With the ruler and triangle, draw a perpendicular to a given line *AB* through a given point *P* without the line (Fig. 26).

50. Draw a straight line *AB* and in it mark a point *D* (Fig. 27). From *D* draw a line forming:

(*a*) Two right angles;

(*b*) One acute angle and one obtuse angle.

51. In Fig. 27, draw two lines from *D* forming:

(*a*) One right angle and two acute angles;

(*b*) One obtuse angle and two acute angles;

(*c*) Three acute angles.

52. From a given point draw three lines forming:

(*a*) Three obtuse angles;

(*b*) Two obtuse angles and one right angle;

(*c*) Two obtuse angles and one acute angle.

53. From a given point draw four lines forming:

(*a*) Two obtuse angles and two acute angles;

(*b*) Two obtuse angles, one right angle, and one acute angle;

(*c*) One obtuse angle, two right angles, and one acute angle;

(*d*) Three obtuse angles and one acute angle;

(*e*) One obtuse angle, one right angle, and two acute angles;

(*f*) One obtuse angle and three acute angles.

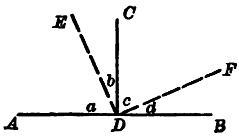


Fig. 28.

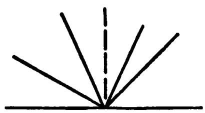


Fig. 29.

54. Draw *CD* perpendicular to *AB* (Fig. 28). Then angles *CDA* and *CDB* are right angles by construction. Draw *DE* and *DF* as in the figure, dividing each right angle into two angles, *a* and *b*, *c* and *d*.

Is the sum of the two angles *a* and *b* equal to the right angle *CDA*?

Is the sum of the two angles *c* and *d* equal to the right angle *CDB*?

Is the whole equal to the sum of all its parts?

The sum of the angles *a*, *b*, *c*, and *d* is equal to how many right angles?

55. What is the sum of all the angles that can be formed at a point in a straight line and on the same side of the line (Fig. 29)?

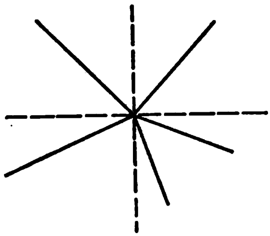


FIG. 30.

56. What is the sum of all the angles that can be formed around a point in a plane (Fig. 30)?

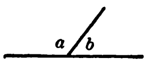


FIG. 31.

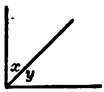


FIG. 32.

Supplementary Angles. Two angles whose sum is equal to two right angles are called **supplementary angles.** Thus, if angles *a* and *b* (Fig. 31) are together equal to two right angles, they are supplementary.

Angle a is the supplement of angle b, and angle b is the supplement of angle a.

The **supplement of an angle** is the difference between that angle and two right angles.

Complementary Angles. Two angles whose sum is equal to one right angle are called **complementary angles.** Thus, if angles x and y (Fig. 32) are together equal to one right angle, they are complementary. Angle x is the complement of angle y, and angle y is the complement of angle x.

The **complement of an angle** is the difference between that angle and a right angle.

57. Draw an angle, and then construct its supplement. (Make your construction lines of short dashes. Thus, – – – –.)

58. Draw an angle, and then construct its complement.

59. Draw an angle, and then construct its supplement and its complement.

60. Which is the greater, the supplement or the complement of a given angle?

61. Draw an acute angle, its supplement, and its complement. Subtract the complement from the supplement. What angle remains?

62. What is the supplement of a right angle?

63. What is the complement of half a right angle?

64. Is the supplement of an acute angle acute or obtuse?

65. Is the supplement of an obtuse angle acute or obtuse?

66. Is the complement of an angle acute or obtuse?

Divisions of the Right Angle. The right angle is divided into 90 equal parts called *degrees* (°); hence every right angle contains 90 degrees (90°).

The degree is divided into 60 equal parts called *minutes* (′).

The minute is divided into 60 equal parts called *seconds* (″).

Thus, 34° 18′ 20″ is read, "Thirty-four degrees, eighteen minutes, twenty seconds."

67. Read the following angles: 32° 55′ 06″; 47° 29′ 50″; 81° 11′ 63″; 128° 49′ 17″.

68. Write the following angles: nine degrees, fourteen minutes, twenty seconds; sixteen degrees, fifty-two minutes, twelve seconds; eighty-four degrees, five minutes, thirty-three seconds.

69. How many degrees in half a right angle?

70. How many degrees in $\frac{2}{3}$ of a right angle?

71. How many degrees in two right angles?

72. How many degrees in four right angles?

73. What part of a right angle is 45°?

74. What part of a right angle is 30°?

75. What part of a right angle is 60°?

76. How many right angles are there in 270°?

77. Reduce 225° to right angles.

78. How many degrees in the supplement of each of the following angles: 160°; 100°; 90°; 84°; 30°?

79. How many degrees in the complement of each of the following angles: 5°; 16°; 30°; 45°; 78°?

80. Find an angle whose supplement is 154° 30′ 20″.

81. Find an angle whose complement is 41° 22′ 58″.

82. What is the greatest number of degrees an obtuse angle can have? the least number?

83. What is the greatest number of degrees an acute angle can have? the least number?

The **Protractor** is an instrument used to measure and to construct angles.

One form of the protractor is shown in Fig. 33, in

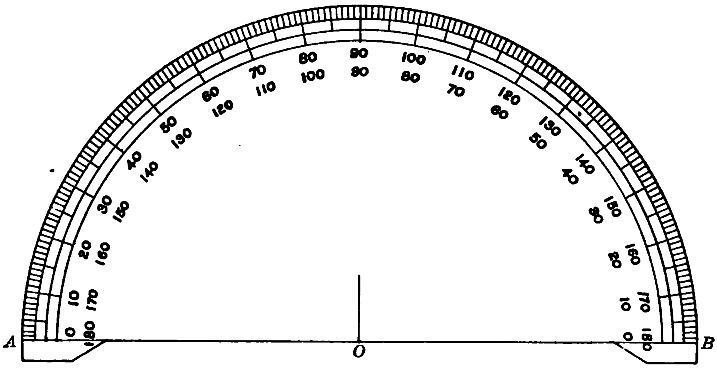


Fig. 33.—The Protractor.

which O, the middle point of AB, is the vertex of the angle. The semicircle is graduated from 0° to 180° from left to right and from right to left.

TO MEASURE A GIVEN ANGLE

To measure an angle with the protractor, place the point O on the vertex of the angle, and the line AB on one side of the angle; then the remaining side of the angle will indicate on the edge of the protractor the size of the angle.

84. Draw five acute angles, measure each with the protractor, and write in each the number of degrees it contains.

85. Draw five obtuse angles, measure each with the protractor, and write in each the number of degrees it contains.

86. With the ruler, and judging by the eye only, draw

angles of 30°, 45°, and 60°. Measure each with the protractor and note the errors.

87. With the ruler draw ten different angles, estimate (by the eye only) the number of degrees in each, then measure each with the protractor, and record your work in a table constructed as follows (Fig. 34):

Angle	DEGREES		Error
	Estimated	Measured	
1	40	38	2
2			
3			
Totals			

FIG. 34.

88. Draw one straight line to meet another, measure the two angles formed, and take their sum.

89. Form any three angles at a point in a line and on the same side of the line (Fig. 35). Measure each and take their sum.

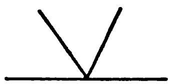


FIG. 35.

Should the answers in 88 and 89 be the same? (Compare 55.)

90. Draw four lines from the same point, measure the four angles formed, and take their sum. Repeat the experiment with five lines. (Compare 56.)

TO CONSTRUCT AN ANGLE OF A GIVEN VALUE

91. Construct an angle of 50°.

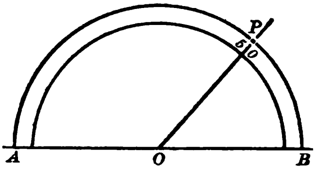


FIG. 36.

Construction. Hold the protractor firmly on the paper and draw the line *OB* (Fig. 36). Then make a dot *P* at the edge of the protractor marking the angle 50°. Remove the protractor and draw *PO*. Then *POB* is an angle of 50°.

With the ruler and protractor construct the following angles:

92. 20°	96. 120°	100. 90°
93. 45°	97. 100°	101. 82°
94. 38°	98. 150°	102. 47°
95. 15°	99. 135°	103. 26°

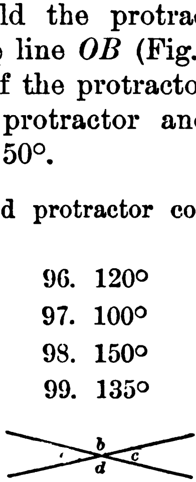


FIG. 37.

Vertical Angles. When two straight lines intersect, the four angles thus formed are called, in pairs, **vertical angles.** Thus (Fig. 37), angles *a* and *c* are vertical angles; also angles *b* and *d*.

104. Draw two intersecting lines, measure the vertical angles thus formed, and write in each the number of degrees

it contains. Repeat the experiment with other vertical angles. Generalize.

105. Draw two lines intersecting at an angle of 60°, and write in each angle the number of degrees it contains.

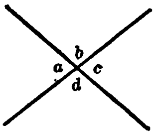


FIG. 38.

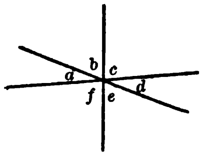


FIG. 39.

106. If (Fig. 38) angle a equals 78°, how many degrees in angle c? in angle b? in angle d?

107. If angle $a = 25°$ (Fig. 39) and angle $b = 70°$, find each of the other angles.

108. If two lines intersect forming one right angle, what is the value of each of the other three angles?

THE PERPENDICULAR BISECTOR OF A LINE

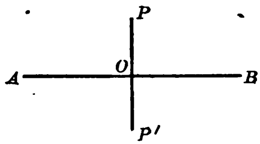


FIG. 40.

If PP' (read P P prime) is perpendicular to AB at the middle point of AB, PP' is called **the perpendicular bisector** of AB (Fig. 40).

Note. *To bisect* means to divide into two equal parts.

The geometric method of constructing the perpendicular bisector of a line is given on page 38.

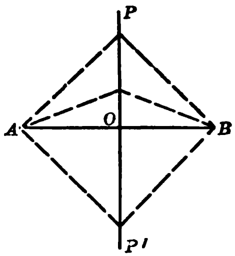


FIG. 41.

109. Draw a line AB (Fig. 41) two inches long and mark its middle point O. Through O draw PP' perpendicular to AB. (Draw the figure with absolute precision.) With the dividers measure the distance of any point in PP' from A and B. Repeat the measurement from other points in PP'. Generalize.

From 109 we derive the following principles which are of great importance in your future work:

110. *Every point in the perpendicular bisector of a line is equidistant from the extremities of the line.*

111. *Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.*

112. Show that the shortest line which can be drawn from a point to a line is the perpendicular from the point to the line (use compasses).

CHAPTER IV

CURVED LINES

So far we have considered *straight* lines only.

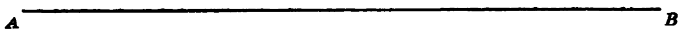


FIG. 42.—A Straight Line.

A **straight line** is a line of unchanging direction; as, *AB* (Fig. 42).

A straight line is usually called a *line* simply.

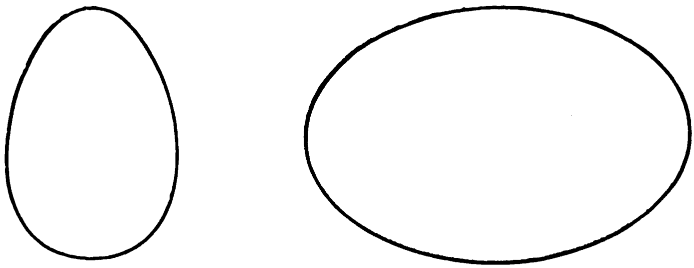


FIG. 43.—Curved Lines.

A **curved line** is a line no part of which is straight (Fig. 43).

A curved line is usually called a *curve*.

A straight line changes its direction at no point; a curved line changes its direction at every point.

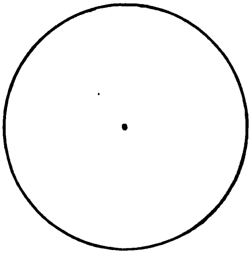


FIG. 44.—The Circle.

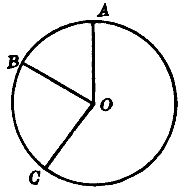


FIG. 45.

The most familiar curve is the *circle* (Fig. 44).

A **circle** is a plain figure bounded by a curved line all points of which are equidistant from a point within called the center.

The bounding line of a circle is called the **circumference.**

The point *O* is called the center of the circle (Fig. 45).

A straight line drawn from the center to the circumference is called a **radius** (plural radii). Thus, *OA* is a radius; also, *OB* and *OC*.

An **arc** is any part of the circumference; as, *AB*, or *BC*.

From the definition of a circle we have the following principles:

113. *All radii of the same circle are equal.*

114. *All radii of equal circles are equal.*

115. Draw two intersecting circles.

116. In how many points can two circles intersect?

In the pupil's future work, 113, 114, and 116 (two circles can intersect in two points only) are of great importance. Observe the application of these principles in the **Solution** and **Proof of Problems** 117 and 120.

117. **Problem.** *To bisect a given straight line.*

Let AB be the given straight line (Fig. 46).

We are required to bisect AB; that is, to divide AB into two equal parts.

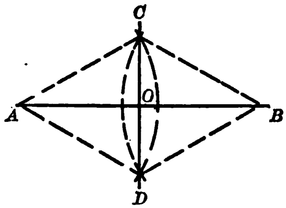


FIG. 46.

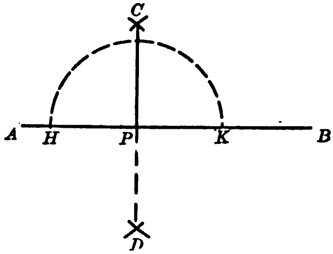


FIG. 47.

Solution. From A and B as centers, and with a radius greater than the half of AB, describe arcs intersecting at C and D.

Draw CD intersecting AB at O.

Then CD bisects AB at O.

Proof. C is equidistant from A and B, for "*All radii of equal circles are equal*" (114).

Likewise D is equidistant from A and B.

Therefore CD bisects AB at O, for "*Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line*" (111).

118. Draw a perpendicular to a given line at its middle point.

119. Draw a perpendicular to a given line at a given point in the line.

(Let AB be the given line and P the given point in AB (Fig. 47). From P as a center, and with any convenient radius, describe an arc cutting AB at H and K. This is the *key* to the solution. You are now to draw a perpendicular to HK at its middle point P, and the problem is the same as 118.)

120. Problem. *To draw a perpendicular to a given line from a given point without the line.*

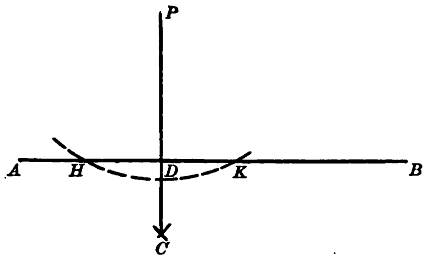


FIG. 48.

Let AB be the given line and P the given point without the line AB (Fig. 48).

We are required to draw a perpendicular from the point P to the line AB.

Solution. From P as a center, and with a radius sufficiently great, describe an arc cutting AB at H and K.

From H and K as centers, and with a radius greater than the half of HK, describe arcs intersecting at C.

Draw PC intersecting AB at D.

Then PD is perpendicular to AB.

Proof. P is equidistant from H and K, for "*All radii of the same circle are equal*" (113).

C is equidistant from H and K, for "*All radii of equal circles are equal*" (114).

Therefore PD is perpendicular to HK, for "*Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line*" (111).

CHAPTER V

TRIANGLES

A **triangle** is a plane figure bounded by three straight lines.

The **sides** of a triangle are the straight lines which bound it.

The **angles** of a triangle are the angles included by the sides, and the vertices of the angles are called the **vertices** of the triangle.

The **perimeter** of a triangle is the sum of its three sides.

TRIANGLES CLASSIFIED WITH RESPECT TO THEIR SIDES

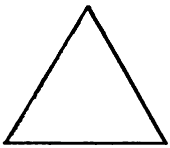


FIG. 49.—Equilateral Triangle.

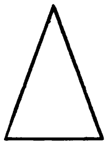


FIG. 50.—Isosceles Triangle.

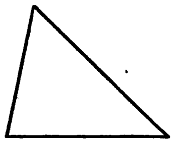


FIG. 51.—Scalene Triangle.

An **equilateral triangle** has its three sides equal (Fig. 49).

An **isosceles triangle** has two sides equal (Fig. 50).

A **scalene triangle** has no two sides equal (Fig. 51).

TRIANGLES CLASSIFIED WITH RESPECT TO THEIR ANGLES

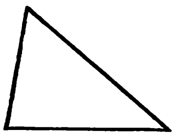


FIG. 52.—Acute Triangle.

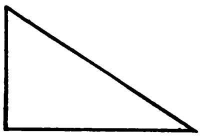


FIG. 53.—Right Triangle.

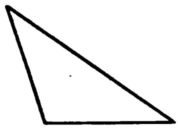


FIG. 54.—Obtuse Triangle.

An **acute triangle** has all its angles acute (Fig. 52).
A **right triangle** has one right angle (Fig. 53).
An **obtuse triangle** has one obtuse angle (Fig. 54).

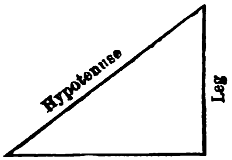


FIG. 55.—Right Triangle.

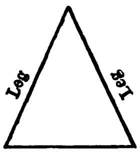


FIG. 56.—Isosceles Triangle.

In a right triangle, the side opposite the right angle is called the **hypotenuse,** and the other two sides are called the **legs** (Fig. 55).

In an isosceles triangle, the equal sides are called the **legs,** and the third side is called the **base** (Fig. 56).

Any side of a triangle may be considered the *base.*

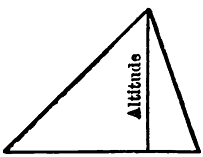


FIG. 57.

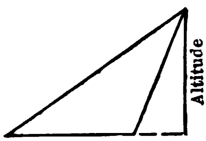


FIG. 58.

The **altitude** of a triangle is the perpendicular drawn from the vertex of the triangle to the base (Fig. 57), or to the base produced (Fig. 58).

The **dimensions** of a triangle are the *base* and *altitude*.

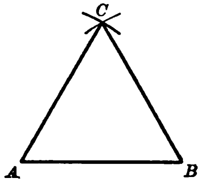


FIG. 59.

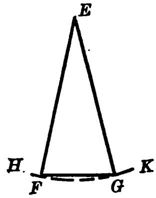


FIG. 60.

121. Construct an equilateral triangle.

(Begin by drawing a line *AB* of any convenient length (Fig. 59). Next, from *A* and *B* as centers, and with *AB* as a radius, describe arcs intersecting in *C*. Now join *CA* and *CB*. Then *CAB* is an equilateral triangle. Why?)

122. Construct an isosceles triangle.

(The following is a good method for constructing an isosceles triangle: From some point *E* strike an indefinite arc *HK* (Fig. 60). Join any two points in the arc, as *F* and *G*. Draw *EF* and *EG*. Then *EFG* is an isosceles triangle. Why?)

123. Construct a scalene triangle.

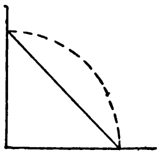


FIG. 61.

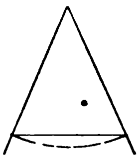


FIG. 62.

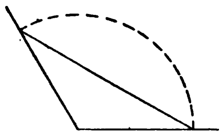


FIG. 63.

124. Construct an isosceles right triangle (Fig. 61).

125. Construct an isosceles acute triangle (Fig. 62).

126. Construct an isosceles obtuse triangle (Fig. 63).

127. Construct a scalene right triangle.

128. Construct a scalene acute triangle.

129. Construct a scalene obtuse triangle.

130. Can you draw an equilateral triangle that is not acute?

131. Can you draw an acute triangle that is not equilateral?

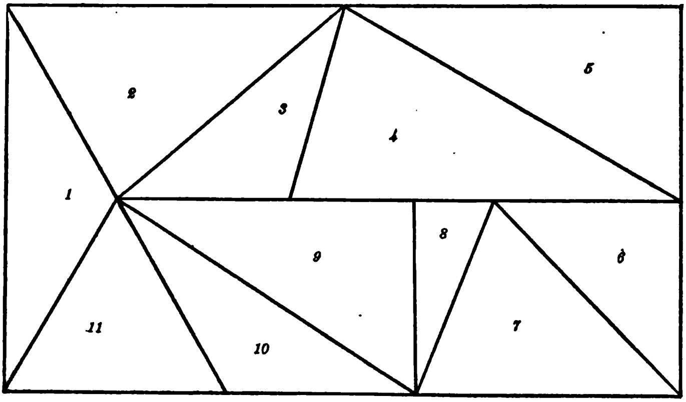


FIG. 64.

132. Describe the triangles in Fig. 64.

133. Draw an isosceles triangle and measure the base angles (angles at the base). Repeat the experiment with other isosceles triangles. Generalize.

134. Draw on transparent paper an isosceles triangle, draw its altitude, and fold along the altitude. Compare the two right triangles.

135. Draw an equilateral triangle and measure its angles. Generalize.

136. Construct a triangle whose sides are 5, 6, 7, and measure its angles. To what class does it belong?

137. Construct a triangle whose sides are 3, 4, 5, and measure its angles. To what class does it belong?

138. Construct a triangle whose sides are 3, 5, 7, and measure its angles. To what class does it belong?

139. Construct an isosceles triangle whose vertical angle is 40°.

140. Construct an isosceles triangle whose vertical angle is 90°.

141. Construct an isosceles triangle whose vertical angle is 120°.

142. Draw the three altitudes of an acute triangle.

143. Draw the three altitudes of an obtuse triangle.

Note. The three altitudes of any triangle intersect in a point.

144. Draw the three altitudes of an equilateral triangle and measure each. Generalize.

145. Construct a right triangle whose legs are 1 inch and $1\frac{1}{2}$ inches.

146. Construct a triangle whose sides are 1 inch, $1\frac{1}{4}$ inches, and $1\frac{1}{2}$ inches.

147. Construct a triangle with two sides 1 inch and $1\frac{1}{2}$ inches respectively, and the included angle 40°.

148. Construct a triangle having one side 2 inches, and the adjacent angles 50° and 60° respectively.

149. Construct a right triangle with one leg $1\frac{1}{4}$ inches and the adjacent acute angle 60°.

150. Construct a right triangle with hypotenuse $1\frac{3}{4}$ inches and one leg 1 inch.

151. Can you construct a triangle with sides 1 inch, 2 inches, and 3 inches?

152. Construct an isosceles triangle whose base is 2 inches and a leg 3 inches.

153. Construct an isosceles triangle whose base is 2 inches and altitude 3 inches. (See 134.)

154. The perimeter of an isosceles triangle is 11 inches. Find the base if a leg is 4 inches.

155. The perimeter of an isosceles triangle is 13 inches. Find a leg if the base is 3 inches.

156. Construct an equilateral triangle whose perimeter is 6 inches.

157. Divide an isosceles triangle into two equal triangles. (See 134.)

158. Divide an equilateral triangle into three equal triangles. (See 144.)

159. Trisect a right angle (with ruler and compasses).

THE SUM OF THE ANGLES OF A TRIANGLE

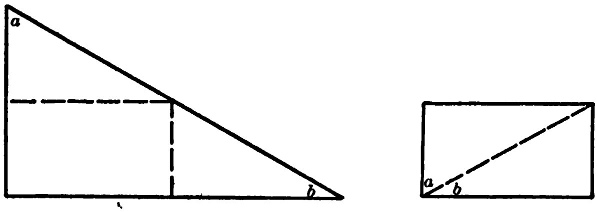
160. Construct an equilateral triangle, measure its angles and take their sum. (See 135.)

161. Construct a right triangle, measure its angles and take their sum.

162. Draw five different scalene triangles, mark the angles in each *A*, *B*, *C*, measure the angles in each, and record your work in a table constructed as follows (Fig. 65):

Triangle	DEGREES IN			Sum of A, B, C
	A	B	C	
1				
2				
3				
4				
5				

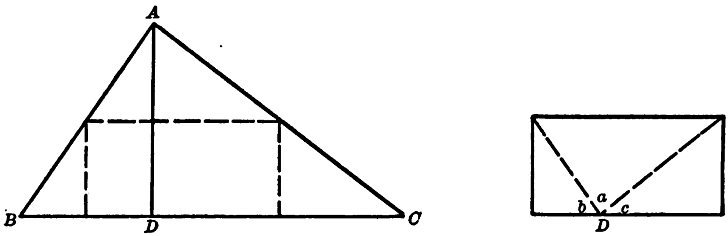
Fig. 65.



Before Folding. After Folding.

FIG. 66.

163. Cut out a right triangle, and fold the vertices of the acute angles upon the vertex of the right angle (Fig. 66), making the two acute angles coincide with the right angle.



Before Folding. After Folding.

FIG. 67.

164. Draw a triangle *ABC*, and draw *AD* perpendicular to *BC*. Cut out the triangle and fold the three vertices upon the point *D* (Fig. 67), making the three angles of the triangle coincide with the two right angles at *D*.

165. What is the sum of the angles of a triangle?

166. If one angle of a triangle is 80°, find the sum of the other two angles.

167. If two angles of a triangle are 52° and 84°, find the third angle.

168. How many right angles can a triangle have?

169. How many obtuse angles can a triangle have?

170. What is the sum of the two acute angles of a right triangle?

171. Is the sum of the two acute angles of an obtuse triangle acute or obtuse?

172. If the acute angles of a right triangle are equal, how many degrees in each?

173. How many degrees in each angle of an equiangular triangle?

174. Construct a right triangle in which one leg is 3 inches and the adjacent acute angle 50°.

175. Construct a right triangle in which the hypotenuse is 2 inches and one angle 75°.

176. If the vertical angle of an isosceles triangle contains 70°, how many degrees in each angle at the base?

177. If a base angle of an isosceles triangle contains 50°, how many degrees in the vertical angle?

178. Construct an isosceles triangle in which the vertical angle is 100°.

179. Construct an isosceles triangle in which a base angle is 78°.

180. Construct an isosceles triangle in which a base angle is twice the vertical angle.

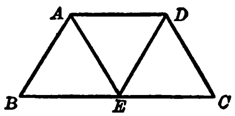


FIG. 68.

181. Figure 68 is formed of three equilateral triangles.

1. How many degrees in angle B? in angle C?
2. How many degrees in angle BAD? in angle ADC?
3. How many degrees in each angle at E?
4. What figure is $ABCD$?
5. What figure is $ABED$? $AECD$?

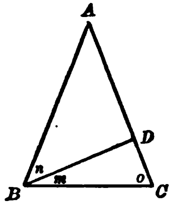


FIG. 69.

182. In Fig. 69, $AB = AC$, angle $A = 50°$, and BD is perpendicular to AC.

1. How many degrees in angle o?
2. How many degrees in angle m?
3. How many degrees in angle n?

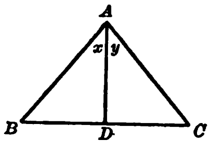


FIG. 70.

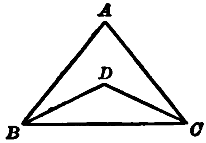


FIG. 71.

183. In the isosceles triangle ABC, AD is perpendicular to the base BC, and angle $B = 50°$ (Fig. 70).

1. How many degrees in angle x?
2. How many degrees in angle y?

184. In the isosceles triangle ABC, DB and DC bisect the angles B and C respectively (Fig. 71). If the angle A contains 88°, how many degrees in angle D?

CHAPTER VI

QUADRILATERALS

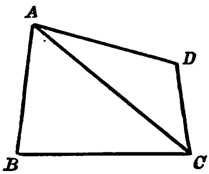


FIG. 72.

A **quadrilateral** is a plane figure bounded by four straight lines (Fig. 72).

A **diagonal** of a quadrilateral is a straight line joining two opposite vertices; as, AC.

185. Draw a quadrilateral, and draw a diagonal.

Into how many triangles does a diagonal divide a quadrilateral?

What is the sum of the angles of each triangle?

What is the sum of the angles of a quadrilateral?

186. Three angles of a quadrilateral are 50°, 70°, 100°. Find the fourth angle.

187. Three angles of a quadrilateral are right angles. What is the value of the fourth angle?

188. Can you construct a quadrilateral having four acute angles?

189. Can you construct a quadrilateral having four obtuse angles?

190. Construct a quadrilateral having two right angles only.

191. Construct a quadrilateral having four right angles.

192. Construct a quadrilateral having two obtuse angles. What kind of angles are the other two?

193. If two angles of a quadrilateral are supplementary, what can you say of the other two angles?

QUADRILATERALS CLASSIFIED

FIG. 73.—Parallelogram.

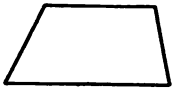


FIG. 74.—Trapezoid.

A **parallelogram** is a quadrilateral whose opposite sides are parallel (Fig. 73).

A **trapezoid** is a quadrilateral which has two sides parallel, and two non-parallel (Fig. 74).

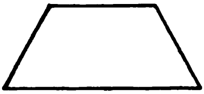


FIG. 75.—Isosceles Trapezoid.

FIG. 76.—Trapezium.

An **isosceles trapezoid** is one whose non-parallel sides are equal (Fig. 75).

A **trapezium** is a quadrilateral which has no two sides parallel (Fig. 76).

PARALLELOGRAMS

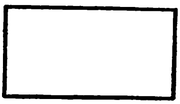


FIG. 77.—Rectangle.

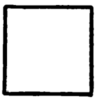


FIG. 78.—Square.

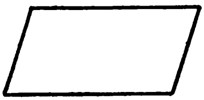


FIG. 79.—Rhomboid.

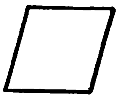


FIG. 80.—Rhombus.

A **rectangle** is a parallelogram whose angles are right angles (Fig. 77).

A **square** is an equilateral rectangle (Fig. 78).

A **rhomboid** is a parallelogram whose angles are oblique (Fig. 79).

A **rhombus** is an equilateral rhomboid (Fig. 80).

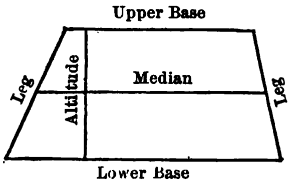


Fig. 81.

The **bases** of a trapezoid are the parallel sides (Fig. 81).

The **legs** of a trapezoid are the non-parallel sides (Fig. 81).

The **median** of a trapezoid is the straight line joining the middle points of the legs (Fig. 81).

The **altitude** of a trapezoid or parallelogram is the perpendicular distance between the bases (Fig. 81).

Any side of a parallelogram may be considered the *base.*

194. Draw the different kinds of parallelograms and upon each write its name.

195. Name the quadrilaterals in Fig. 82.

196. In what do the square and the rectangle agree? in what do they differ?

197. In what do the square and the rhombus agree? in what do they differ?

198. In what do the rhombus and the rhomboid agree? in what do they differ?

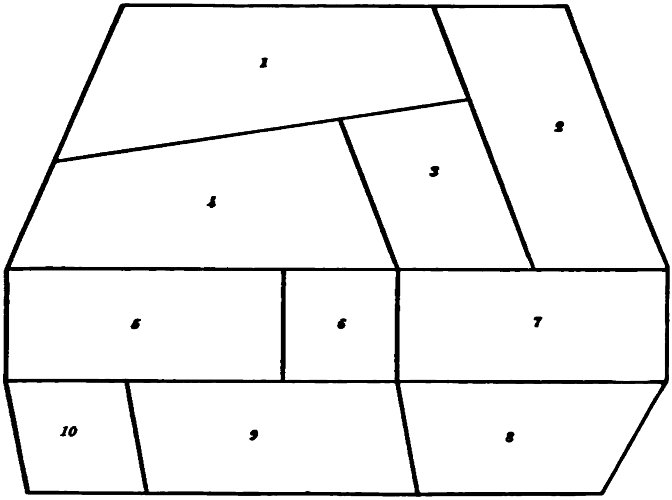


FIG. 82.

199. Measure the opposite sides of different parallelograms. Generalize.

200. Measure the opposite angles of different parallelograms. Generalize.

201. Draw a parallelogram (rhomboid), cut out, cut along a diagonal, and superpose the triangles. Generalize.

202. Measure the segments of the diagonals of the different classes of parallelograms. Generalize.

203. Measure the angles between the diagonals of a square; of a rhombus. Generalize.

204. Measure the diagonals of a rectangle; of a square. Generalize.

205. Measure the angles which the diagonals of a square make with adjacent sides; the same for the rhombus. Generalize.

206. One angle of a parallelogram is 80°. Find the other angles.

Construct the following parallelograms, and on each write its name:

207. One angle 90°, adjacent sides 2 inches.

208. One angle 90°, adjacent sides 2 and 3 inches.

209. One angle 68°, adjacent sides $2\frac{1}{2}$ inches.

210. One angle 110°, adjacent sides $1\frac{1}{2}$ and $2\frac{1}{2}$ inches.

211. What must be given to find the perimeter of a square?

212. What must be given to find the perimeter of a rhombus?

213. What must be given to find the perimeter of a rectangle?

214. What must be given to find the perimeter of a rhomboid?

215. What must be given to find the perimeter of a trapezoid?

216. What must be given to find the perimeter of a trapezium?

217. Draw a trapezoid with two equal sides.

218. Draw a trapezium with two equal sides.

219. Draw a trapezium with three equal sides.

220. Can you draw a trapezium with four equal sides?

221. Construct a square having a perimeter of 6 inches.

222. Construct a rectangle having a perimeter of 10 inches and a side 2 inches.

223. Construct a rhomboid having a perimeter of 7 inches, one side $2\frac{1}{2}$ inches, and one angle 80°.

224. Construct a rhombus having a perimeter of 9 inches and an angle 100°.

225. Construct a square whose diagonal is 2 inches. (See 202, 203, 204.)

226. Construct a parallelogram with one diagonal $2\frac{1}{2}$ inches, and whose sides are 2 and 1 inches.

227. Construct a rectangle whose base is 2 inches and whose diagonal is $2\frac{1}{4}$ inches.

228. Construct a rhombus with a side 2 inches, and one diagonal $2\frac{1}{4}$ inches.

229. Construct a rhombus with one diagonal equal to a side.

230. Construct an isosceles trapezoid with base 2 inches, altitude 1 inch, and a leg $1\frac{1}{8}$ inches.

231. Construct two squares so that one angle in each may be vertical angles.

232. Construct a square, and on each side construct an equilateral triangle.

233. Construct an equilateral triangle, and on each side construct a square.

234. Divide a square into four equal squares.

235. Divide a square into four equal triangles.

236. Divide a square into four equal trapeziums.

237. How many equal squares may touch at one point?

238. How many equal squares may be placed around an equal square so as to touch it?

239. Divide a parallelogram into two equal parts in four different ways.

CHAPTER VII

POLYGONS IN GENERAL

A **polygon** is a plane figure bounded by straight lines.

A **triangle** is a polygon of three sides.

A **quadrilateral** is a polygon of four sides.

A **pentagon** is a polygon of five sides.

A **hexagon** is a polygon of six sides.

An **octagon** is a polygon of eight sides.

A **decagon** is a polygon of ten sides.

A **dodecagon** is a polygon of twelve sides.

A **diagonal** of a polygon is a straight line joining any two vertices not adjacent (Fig. 83).

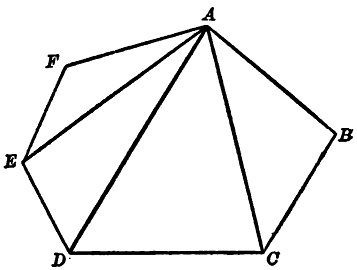


FIG. 83.

240. Name the diagonals in Fig. 83.

241. Can you draw a diagonal in a triangle?

242. How many diagonals can you draw from one vertex

in a quadrilateral? in a pentagon? in a hexagon? in an octagon?

How does the number of diagonals compare with the number of sides of the polygon in each case?

243. How many diagonals can you draw from all the vertices in a quadrilateral? in a pentagon? in a hexagon? in an octagon?

244. Can you devise a rule for finding the number of different diagonals which may be drawn in any polygon?

245. How many different diagonals can be drawn in a decagon? in a dodecagon?

246. Into how many triangles can a quadrilateral be divided by diagonals? a pentagon? a hexagon? an octagon?

How does the number of triangles compare with the number of sides in each case?

247. What is the sum of the angles of a pentagon?

248. What is the sum of the angles of a hexagon?

249. What is the sum of the angles of an octagon?

250. What is the sum of the angles of a decagon?

251. What is the sum of the angles of a dodecagon?

252. Can you write a rule for finding the sum of the angles of any polygon?

CHAPTER VIII

AREAS

253. Draw a line one inch long and construct a square upon it.

Such a square is called a *square inch*; it is a *unit of measure for surfaces.* For example, the problem "How many square inches in a sheet of paper 8 inches long and 5 inches wide?" means how many square inches will it take to cover the sheet of paper?

254. Draw a rectangle 8 inches long and 5 inches wide and divide it into square inches. How many rows of squares are there? How many squares in each row? How many square inches in the area?

How do you find the area of a rectangle?

The **dimensions** of a rectangle are its base and altitude.

255. What is the area of a rectangle whose base is 30 feet and altitude 20 feet?

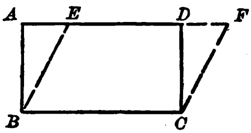


FIG. 84.

256. The rectangle *ABCD* and the parallelogram *EBCF* have the same base *BC* and the same altitude *CD* (Fig. 84).

Can you show that the rectangle and the parallelogram have equal areas?

How do you find the area of a rectangle?

How do you find the area of a parallelogram?

257. What is the area of a parallelogram whose base is 5 inches and altitude 3 inches?

258. The area of a parallelogram is 28 square inches. If the base is 7 inches, what is the altitude?

259. The area of a parallelogram is 45 square inches. If the altitude is 5 inches, what is the base?

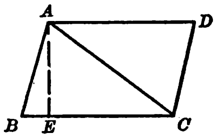


FIG. 85.

260. In 201 you learned that a diagonal divides a parallelogram into two equal triangles.

The parallelogram *ABCD* and the triangle *ABC* have the same base *BC* and the same altitude *AE* (Fig. 85).

The triangle *ABC* is what part of the parallelogram *ABCD*?

How do you find the area of a parallelogram?

How do you find the area of a triangle?

The **dimensions** of a triangle are its base and altitude.

261. Find the area of a triangle whose base is 8 feet and altitude 6 feet.

262. The area of a triangle is 42 square feet. What is its altitude if its base is 12 feet?

263. The area of a triangle is 35 square feet. What is its base if its altitude is 7 feet?

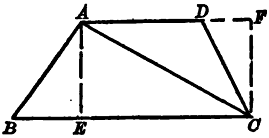


FIG. 86.

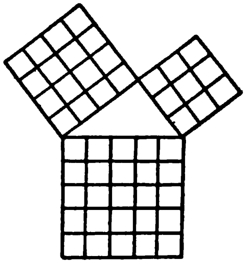


FIG. 87.

264. In the trapezoid $ABCD$ (Fig. 86), AE is perpendicular to BC, and CF is perpendicular to AD produced. Therefore $AECF$ is a rectangle, and $AE = CF$ (199).

The diagonal AC divides the trapezoid into the two triangles ABC and ADC.

The dimensions of the triangle ABC are BC and AE.

The dimensions of the triangle ADC are AD and CF or AE.

The area of the triangle $ABC = \frac{1}{2}(BC \times AE)$.

The area of the triangle $ADC = \frac{1}{2}(AD \times AE)$.

Adding the two equalities, we have

Area of the trapezoid $ABCD = \frac{1}{2}(BC + AD) \times AE$.

Can you now write the rule for finding the area of a trapezoid?

265. Find the area of a trapezoid whose bases are 8 inches and 6 inches, and whose altitude is 5 inches.

266. A rectangle and a rhomboid have the same base and altitude. Which has the less perimeter?

267. Show by a figure that the square on the diagonal of a square is twice the given square.

268. Show by a figure (Fig. 87) that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs.

269. The legs of a right triangle are 6 inches and 8 inches. Find the hypotenuse.

270. The legs of a right triangle are 5 and 12. Find the hypothenuse.

271. The hypothenuse of a right triangle is 25, and one leg is 7. Find the other leg.

272. The hypothenuse of a right triangle is 61, and one leg is 60. Find the other leg.

273. The area of a square is 81 square inches. Find the side.

274. Find the perimeter of a square whose area is 49 square inches.

275. Can you construct a square whose perimeter and area are expressed by the same digit?

CHAPTER IX

CIRCLES

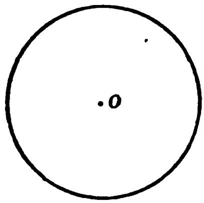


FIG. 88.—A Circle.

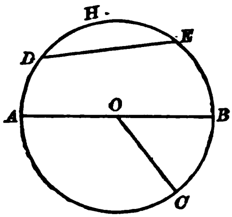


FIG. 89.

A **circle** is a plane figure bounded by a curved line, all points of which are equidistant from a point within called the center (Fig. 88).

The **circumference** of a circle is the curved line which bounds the circle.

The point O is the **center** of the circle.

The **radius** (plural radii) of a circle is a straight line drawn from the center to the circumference; as, OC (Fig. 89).

The **diameter** of a circle is a straight line drawn through the center and terminated by the circumference; as, AB (Fig. 89).

A **chord** of a circle is a straight line which has its extremities in the circumference; as, DE (Fig. 89).

An **arc** is any part of the circumference; as, *DHE* (Fig. 89).

A **semicircle** is half a circle.

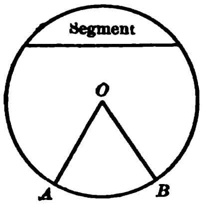


FIG. 90.

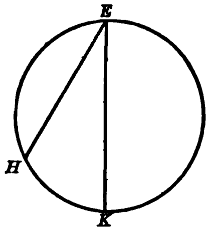


FIG. 91.

A **segment** of a circle is a part of a circle bounded by a chord and its arc (Fig. 90).

An **angle at the center** of a circle is an angle formed by two radii; as, angle *AOB* (Fig. 90).

An **inscribed angle** is an angle whose vertex is in the circumference and whose sides are chords; as, angle *HEK* (Fig. 91).

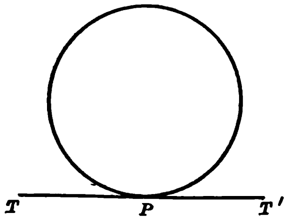


FIG. 92.

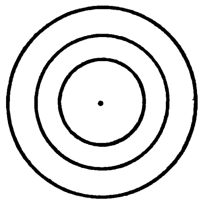


FIG. 93.—Concentric Circles.

A **tangent** to a circle is a straight line of unlimited length which touches the circumference at one point only; as, *TT'* (Fig. 92).

The **point of tangency** is the point in which the tangent touches the circumference; as, *P* (Fig. 92).

276. Draw two circles with the same radius. Are the two circles equal?

277. Are all radii of the same circle or of equal circles equal?

278. Are all diameters of the same circle or of equal circles equal?

279. Draw a circle with a radius one inch long. How long is its diameter?

280. Describe a circle and in it draw a chord equal to the radius (use compasses).

281. Describe a circle and in it draw a chord equal to twice the radius.

282. Show that a diameter bisects a circle and its circumference. (Draw a circle on transparent paper and fold along a diameter. Many of the properties of the circle may be shown by folding and by superposing. When comparing two circles, draw one on transparent paper and slip it over the other.)

Concentric circles are circles which have the same center (Fig. 93).

283. Draw two concentric circles and draw a chord in one tangent to the other.

284. Describe two circles so that each has its center on the circumference of the other.

285. Describe a circle and divide it into two segments.

286. Describe a circle and in it draw two perpendicular diameters.

287. Describe a circle and in it inscribe an acute angle; a right angle; an obtuse angle.

288. Describe two circles intersecting each other. In how many points can two circles intersect?

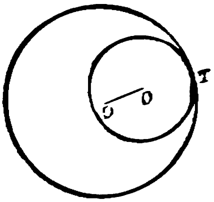


FIG. 94.—Internal Tangency.

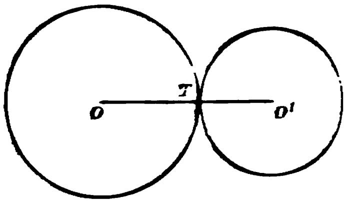


FIG. 95.—External Tangency.

Two circles are **tangent** when they touch each other at one point only (Fig. 94 and Fig. 95).

289. **Draw two tangent circles so that one is within the other.**

290. **Draw two tangent circles so that each is without the other.**

291. **May externally tangent circles be equal? unequal?**

292. **May internally tangent circles be equal? unequal?**

293. **Draw two circles so that each is wholly without the other.**

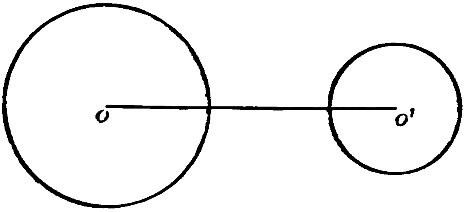


FIG. 96.

The **line of centers** of two circles is the straight line which joins their centers; as OO' (Fig. 96). Also, OO' in Fig. 94 and Fig. 95.

294. **Draw two tangent circles and their line of centers. Observe that the line of centers passes through the point of tangency.**

295. Draw two circles so that their line of centers is equal to the sum of their radii.

296. Draw two circles so that their line of centers is equal to the difference of their radii.

297. Draw two circles so that their line of centers is greater than the sum of their radii.

298. Draw two circles so that their line of centers is less than the difference of their radii.

299. Draw two circles so that their line of centers is zero.

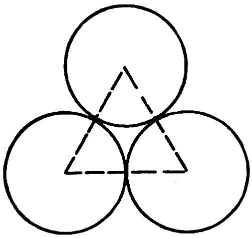


FIG. 97.

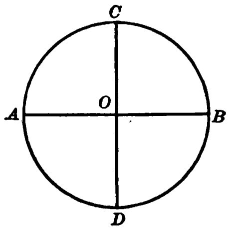


FIG. 98.

300. Draw three equal circles so that each is tangent to the other two (Fig. 97). (The lines of centers of such circles form an equilateral triangle. Why? Hence begin by constructing an equilateral triangle.)

301. Draw four equal circles so that each is tangent to two of the others. (The lines of centers form what polygon?)

302. Draw four circles so that each is tangent to the other three.

If two perpendicular diameters are drawn in a circle they will divide the circumference into four arcs. It may easily be shown by folding, or by superposition, that these four arcs are equal. Each arc, therefore, is one fourth of the circumference and is called a **quadrant**.

A quadrant is divided into 90 equal parts called

degrees. But a right angle is divided into 90 equal parts called degrees. Hence the number of *arc* degrees in the arc AC is equal to the number of *angle* degrees in the angle AOC (Fig. 98).

A central angle of one degree intercepts on the circumference an arc of one degree; a central angle of ten degrees intercepts an arc of ten degrees, etc. In other words, the numerical measure of a central angle is equal to the numerical measure of its intercepted arc. This important fact is concisely stated as follows:

303. *A central angle is measured by its intercepted arc.*

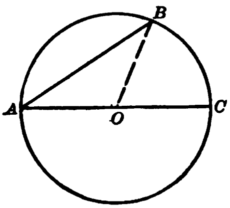


FIG. 99.

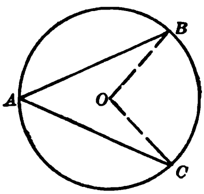


FIG. 100.

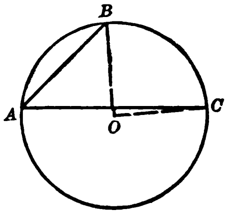


FIG. 101.

304. An inscribed angle is measured by half its intercepted arc. (Let BAC be an angle inscribed in a circle whose center is O. In Fig. 99, O is in one side of the angle. In Fig. 100, O is within the angle. In Fig. 101, O is without the angle. In each case you are to show that the number of degrees in the angle BAC is half the number of degrees in the arc BC, or, which is the same thing, half the number of degrees in the central angle BOC. Show this in each figure by measuring the angles BAC and BOC with the protractor.)

305. Show that angles inscribed in the same segment are equal. (In Fig. 102, prove angles C and D equal by 304.)

306. Show that an angle inscribed in a semicircle is a right

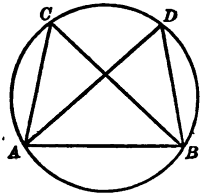


FIG. 102.

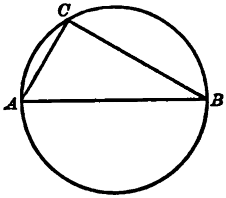


FIG. 103.

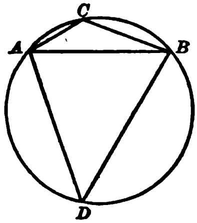


FIG. 104.

angle. (In Fig. 103, the angle C is inscribed in a semicircle. Prove by 304.)

307. Show that an angle inscribed in a segment less than a semicircle is obtuse. (See angle ACB in Fig. 104.)

308. Show that an angle inscribed in a segment greater than a semicircle is acute. (See angle ADB in Fig. 104.)

309. Draw a chord in a circle and inscribe an angle in each segment. What is the sum of the two angles? Why?

310. Draw a perpendicular to a given line at its extremity.

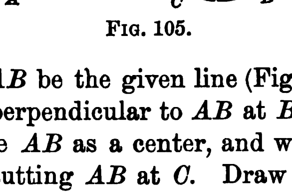


FIG. 105.

Solution. Let AB be the given line (Fig. 105). We are required to draw a perpendicular to AB at B. From any point O without the line AB as a center, and with OB as a radius, describe a circle cutting AB at C. Draw CO and produce it to meet the circumference at D. Draw DB. Then DB is perpendicular to AB.

Proof. The angle DBC is a right angle, for it is inscribed in a semicircle (306). Therefore DB is perpendicular to AB.

311. Construct a right triangle. From the middle point of the hypotenuse as a center, and with half the hypotenuse as a radius, describe a circle. Why does the circumference pass through the vertex of the right angle?

312. Show that a tangent is perpendicular to the radius drawn to the point of tangency. (Use protractor.)

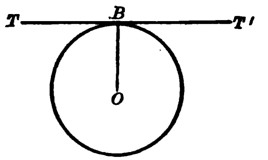


FIG. 106.

313. Draw a tangent to a circle from a given point in the circumference. (In Fig. 106, let B be the given point. Draw the radius OB, then draw TT' perpendicular to OB at B.)

314. Draw a tangent to a circle from a given point without the circle.

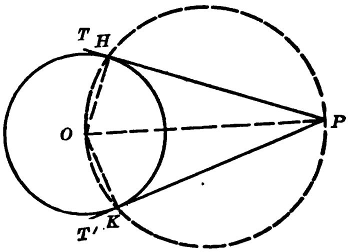


FIG. 107.

Solution. Let O be the center of the circle, and P a given point.

We are required to draw a tangent from P to the circle.

Draw OP. On OP as a diameter describe a circle intersecting the given circle at H and K.

Draw PH, PK, OH, and OK.

PH and PK are the tangents required.

Proof. The angles *OHP* and *OKP* are right angles, for each is inscribed in a semicircle. (306.) Therefore *PH* and *PK* are tangents. (312.)

315. In Fig. 107, show that the tangents *PH* and *PK* are equal; also, that the angles *OPH* and *OPK* are equal. (Fold along the line *PO*.)

316. Show that a diameter perpendicular to a chord bisects the chord and its arc. (Draw figure on transparent paper and fold along the diameter.)

317. Draw two chords in a circle and draw a perpendicular to each at its middle point. Where do the perpendiculars meet? (Take a hint from 316.)

318. Find the center of a given circle. (Take a hint from 317.)

319. Find the center of a given arc. (The solution is the same as that for 318.)

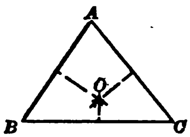


FIG. 108.

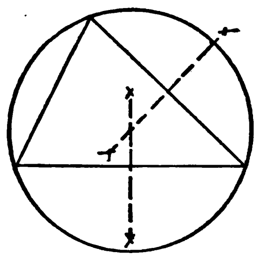


FIG. 109.

320. Draw the perpendicular bisectors of the three sides of a triangle. (The perpendicular should meet in a point. See Fig. 108.)

In Fig. 108 the three perpendicular bisectors of the sides of the triangle meet in the point *O*. By 110 the point *O* is equidistant from *A*, *B*, and *C*, the three vertices of the triangle. This is the key to the solution of the following problem:

321. Circumscribe a circle about a given triangle (Fig. 109). (It is sufficient to bisect two sides only. Why?)

322. Draw a circle through three points not in a straight line. (Join the points, two by two, forming a triangle, and proceed as in 321.)

323. Bisect a given angle.

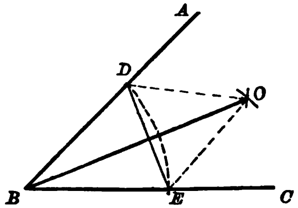


FIG. 110.

Solution. Let *ABC* be a given angle (Fig. 110).

We are required to bisect the angle *ABC*.

From *B* as a center, and with any convenient radius, describe an arc cutting the sides of the angle at *D* and *E*.

From *D* and *E* as centers, and with a radius sufficiently great, describe arcs intersecting at *O*. Join *OB*. Then *OB* bisects the angle *ABC*.

Proof. *OB* bisects the chord *DE* at right angles (111).

Therefore *OB* bisects the arc *DE* (316).

Therefore *OB* bisects the angle *ABC*, for *a central angle is measured by its intercepted arc* (303).

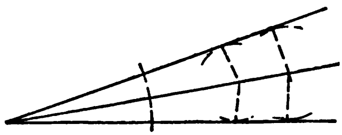


FIG. 111.

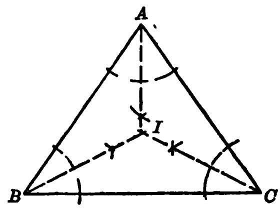


FIG. 112.

324. Show that all points in the bisector of an angle are equidistant from the sides of the angle (Fig. 111). (Draw the

bisector of the angle, as in 323, and with the compasses measure the distance of any point in the bisector from the sides.)

325. Bisect the three angles of a triangle. (The three angle-bisectors should meet in a point. See Fig. 112.)

In Fig. 112 the three angle-bisectors meet in the point *I*. By 324, *I* is equidistant from *AB*, *AC*, and *BC*, the three sides of the triangle.

This is the key to the solution of the following problem:

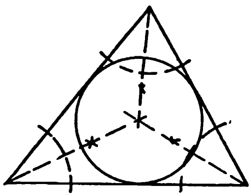


FIG. 113.

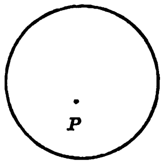


FIG. 114.

326. Inscribe a circle in a triangle (Fig. 113). (It is sufficient to bisect two angles only. Why?)

327. Inscribe a circle in an equilateral triangle.

328. Circumscribe a circle about an equilateral triangle.

329. Show by superposition that in the same circle or in equal circles equal arcs are subtended by equal chords.

330. Mark a point *P* within a circle and draw through it the longest and the shortest possible chords (Fig. 114).

331. On a given chord as base inscribe an isosceles triangle having its vertex in the circumference. (Two solutions. Your figure for 330 will suggest the solution.)

CHAPTER X

REGULAR POLYGONS

A **regular polygon** is a polygon which is equilateral and equiangular.

The *square* and *equilateral triangle* are regular polygons.

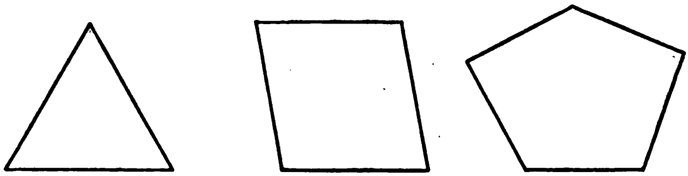


FIG. 115.—Equilateral Polygons.

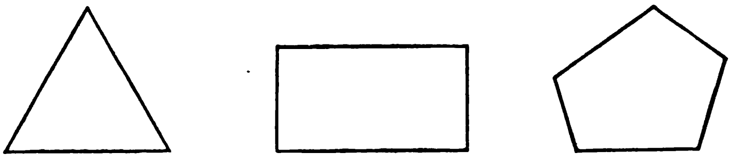


FIG. 116.—Equiangular Polygons.

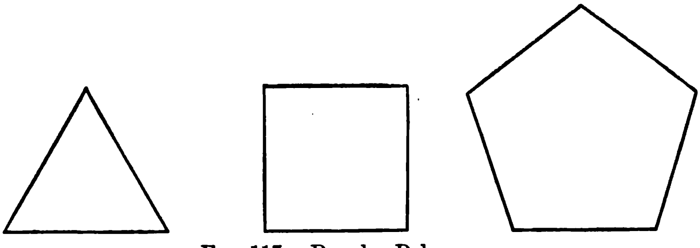


FIG. 117.—Regular Polygons.

Regular polygons are easily constructed by aid of the circle. The following principles are applied:

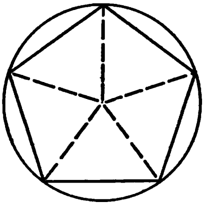


FIG. 118.

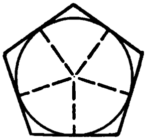


FIG. 119.

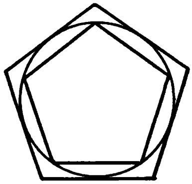


FIG. 120.

Principle 1. If the circumference of a circle be divided into equal arcs, the chords of these arcs will be equal (329), and the angles included by the chords will be equal (305). The polygon thus formed is equilateral and equiangular, and hence regular (Fig. 118).

Principle 2. If the circumference of a circle be divided into equal arcs, the tangents drawn at the points of division will form a regular circumscribed polygon (Fig. 119).

Principle 3. Tangents drawn parallel to the sides of a regular inscribed polygon form a regular circumscribed polygon (Fig. 120).

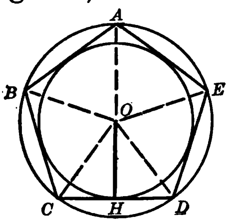


FIG. 121.

The **center** of a regular polygon is the common center of the inscribed and circumscribed circles; as, *O* (Fig. 121).

The **angle at the center** of a regular polygon is the angle formed by radii drawn to the end-points of any side; as, angle AOB.

The **radius** of a regular polygon is the radius of the circumscribed circle; as, OA.

The **apothem** of a regular polygon is the radius of the inscribed circle; as, OH.

In Fig. 121 it will be observed that the five angles about O are equal (*a central angle is measured by its intercepted arc*). Hence the angle $AOB = 360° \div 5 = 72°$. Therefore, to construct a regular pentagon, describe a circle and with the protractor and ruler draw radii forming at the center of the circle angles of 72° each. These radii divide the circumference into five equal arcs whose equal chords form a regular pentagon.

Find the number of degrees in the angle at the center of the following regular polygons:

332. Triangle.
333. Square.
334. Pentagon.
335. Hexagon.
336. Octagon.
337. Decagon.

Inscribe in a circle the following regular polygons:

338. Triangle.
339. Square.
340. Pentagon.
341. Hexagon.
342. Octagon.
343. Decagon.

Circumscribe about a circle the following regular polygons:

344. Triangle.
345. Square.
346. Pentagon.
347. Hexagon.
348. Octagon.
349. Decagon.

350. How many degrees in each angle of a regular pentagon? hexagon? octagon?

351. Inscribe a circle in a square.

352. Circumscribe a circle about a square.

353. Inscribe the largest circle possible in a semicircle.

354. Show that a side of an inscribed regular hexagon is equal to the radius.

355. Show that an inscribed equilateral triangle is formed by joining the alternate vertices of an inscribed regular hexagon.

356. Divide a regular hexagon into six equal triangles.

357. Cut off the angles of an equilateral triangle so as to form a regular hexagon.

358. Cut off the angles of a square so as to form a regular octagon.

359. Construct a regular six-pointed star.

360. Construct a regular five-pointed star.

361. Show by a figure the ratio of the circumscribed and inscribed squares.

362. Show by a figure the ratio of the circumscribed and inscribed equilateral triangles.

363. Show by drawings what regular polygons, or what combinations of regular polygons, may be used in laying tiles.

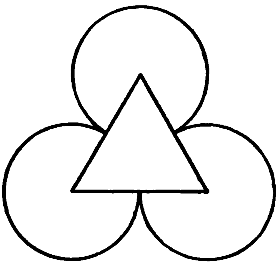


FIG. 122.

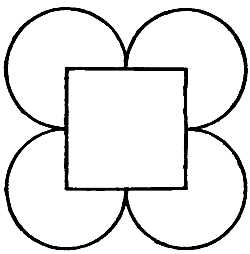


FIG. 123.

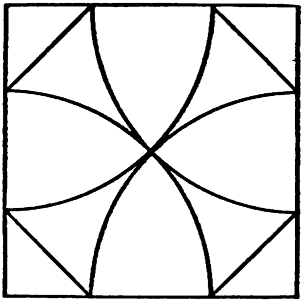


Fig. 124.

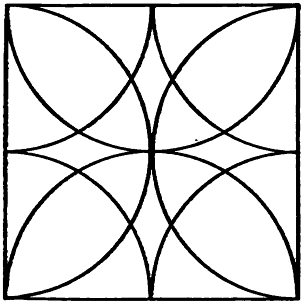


Fig. 125.

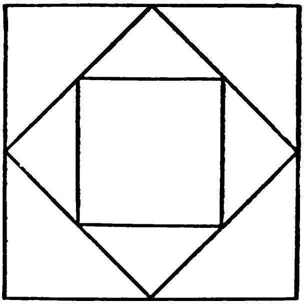


Fig. 126.

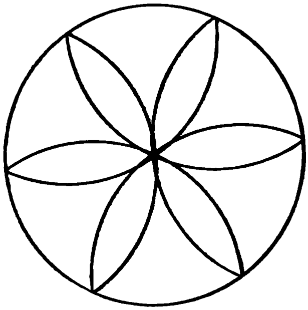


Fig. 127.

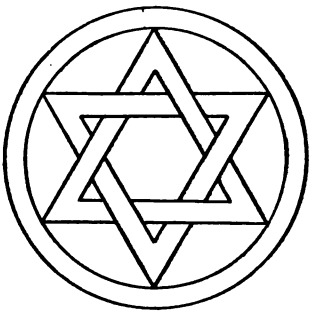


Fig. 128.

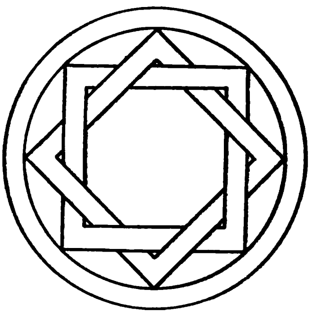


Fig. 129.

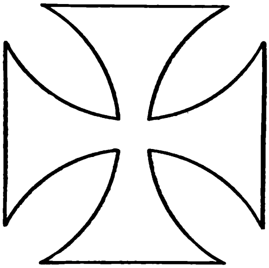


FIG. 130.

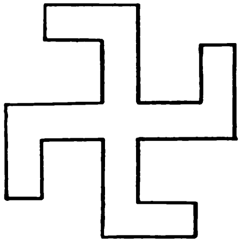


FIG. 131.

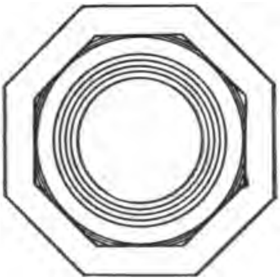


FIG. 132.

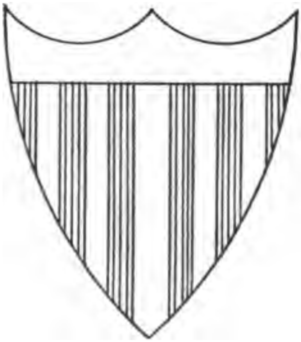


FIG. 133.

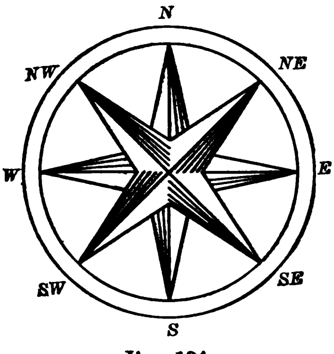


FIG. 134.

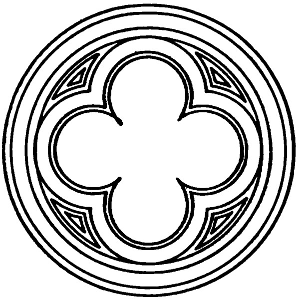


FIG. 135.

CHAPTER XI

AREA OF THE CIRCLE

364. Cut out of stiff cardboard a circle ten inches in diameter and measure its circumference with the greatest possible precision.

What is the ratio of the circumference of a circle to its diameter?

Mathematicians have shown that the exact ratio of the circumference of a circle to its diameter cannot be expressed in numbers, and hence they denote this ratio by the Greek letter π (pi). Therefore the circumference of a circle equals π times the diameter.

Denoting the circumference by C and the diameter by D, we have the convenient formula $C = \pi D$.

For ordinary purposes you may make $\pi = 3\frac{1}{7}$; for greater precision, make $\pi = 3.1416$.

365. Find the circumference of a circle whose diameter is 14 inches.

366. Find the circumference of a circle whose radius is 5 inches.

367. Find the diameter of a circle whose circumference is 66 feet.

368. Find the radius of a circle whose circumference is 100 inches.

369. If the driving wheel of a locomotive engine is 6 feet in diameter, how many revolutions does it make in going one mile?

370. The earth travels around the sun in 365 days. The sun is 93,000,000 miles from the earth. How far does the earth travel in one second?

371. If the radius of a circle is 25 inches, find the length of an arc of 60°. (The arc of 60° is what part of the circumference?)

372. If the diameter of a circle is 21 inches, find the length of an arc of 45°.

Geometers have proved that the area of a circle equals πR^2, in which R is the radius.

Denoting the area of a circle by A, we have the convenient formula $A = \pi R^2$.

373. Find the area of a circle whose radius is 2 inches.

374. Find the area of a circle whose diameter is 20 feet.

375. Find the diameter of a circle whose area is 100 square feet.

376. Find the radius of a circle whose area is one acre.

377. What is the area of the circular ring bounded by two concentric circumferences whose radii are 5 inches and 8 inches?

378. Find the area of the largest circle which can be cut from a cardboard 12 inches square.

379. Find the diameter of a circle equivalent to a square whose side is 6 inches.

380. The radius of a circle is 5 inches. Find the radius of a circle twice as large.

381. Find the area of a semicircle whose radius is 21 inches.

382. The sides of a rectangle are 6 and 8 inches. Find the area of the circumscribed circle.

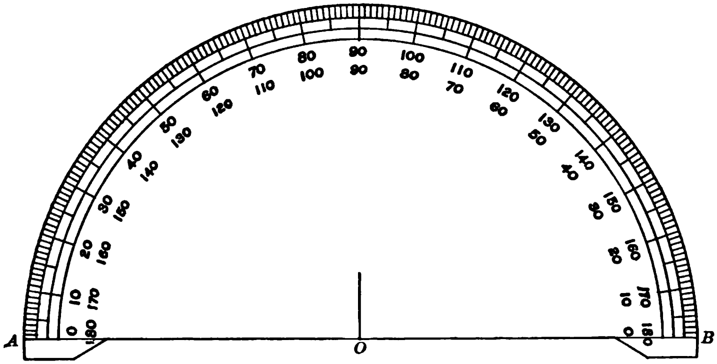
383. Find the side of a square equivalent to a circle whose radius is 11 inches.

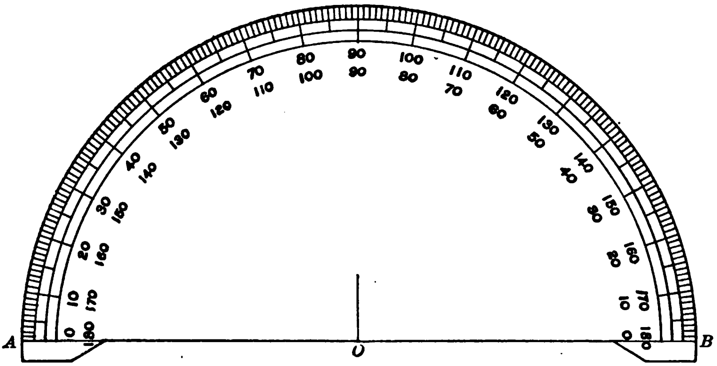
384. Find the area of a circle circumscribing a right triangle whose legs are 20 and 21 inches.

385. A circle is circumscribed about a square whose side is 16 inches. Find the area of one of the segments of the circle cut off by a side of the square.

386. The side of a square is 5 inches, and upon each side as a diameter a semicircle is described within the square. Find the area of one of the leaves formed by the intersection of two semicircles.

INDEX





These protractors may be cut out and used by the pupil in his work.

382. The sides of a rectangle are 6 and 8 inches. Find the area of the circumscribed circle.

383. Find the side of a square equivalent to a circle whose radius is 11 inches.

384. Find the area of a circle circumscribing a right triangle whose legs are 20 and 21 inches.

385. A circle is circumscribed about a square whose side is 16 inches. Find the area of one of the segments of the circle cut off by a side of the square.

386. The side of a square is 5 inches, and upon each side as a diameter a semicircle is described within the square. Find the area of one of the leaves formed by the intersection of two semicircles.